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Note on slightly unstable nonlinear wave systems

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The asymptotic solution for large time of the initial-value problem for weakly nonlinear wave systems is obtained by the method of matched asymptotic expansions in the case in which the linearized problem is slightly unstable. The linearized solution is valid until its small exponential growth overcomes the algebraic decay due to the dispersion of the initial energy. For larger times the nonlinear terms become important, but there are no additional dispersive or diffusive effects. For the non-diffusive case an exact solution which enables the explicit verification of the asymptotic results is found.

1. Introduction

In recent papers by Newell & Whitehead (1971), Stewartson & Stuart (1971) and Hocking, Stewartson & Stuart (1972), the amplitude equation

$$A_t = \epsilon A + kA|A|^2 \tag{1.1}$$

governing the temporal evolution of a spatially periodic unstable wave, originally conjectured by Landau (1944) and later developed by Stuart (1960), was generalized to include the more realistic dispersive and diffusive effects of spatial modulations of the wave envelope due to a relatively arbitrary initial disturbance. The simpler case of propagation of a wave envelope when there is no amplification or decay was derived for a general non-diffusive system by Benney & Newell (1967), and particular solutions were obtained. Stewartson & Stuart (1971) and Hocking *et al.* (1972) were concerned with plane Poiseuille flow when the Reynolds number R was greater than the critical value R_c, such that a small disturbance would be amplified. In §2 the more general amplitude equation is derived for a large class of weakly nonlinear wave systems with small amplification when there is direct modal transfer, a result previously pointed out by Newell & Whitehead (1971).

Stewartson & Stuart (1971) posed the relevant initial-value problem, but did not solve it. Hocking *et al.* (1972) investigated the evolution equation in a simplified case allowing only diffusion. Guided by numerical results, they found that the amplitude first could become singular at some time at one position, a phenomenon they referred to as a 'nonlinear instability burst'. Furthermore, they and Hocking & Stewartson (1972), in the general case, made an attempt at analytically obtaining the structure of the singularity.

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The initial energy of the wave systems discussed here disperses, decreasing the amplitude of the wave envelope from that expected owing to the growth of the slightly unstable wave. Thus it is shown in § 3 that the envelope equation is weakly nonlinear if it is to match to a problem in which the initial energy is not concentrated in the most unstable wavenumber. This suggests that the weakly nonlinear envelope equation is the appropriate one rather than the fully nonlinear equation studied by Hocking *et al.* (1972) and Hocking & Stewartson (1972). The equation and matching conditions governing the spatial and temporal modulations of the wave envelope for large time are solved by asymptotic techniques. Both dispersive and diffusive effects are included. This theory may yield either of two possible terminal states depending on the effects of the nonlinearity. The amplitude of the wave envelope reaches an equilibrium value or, as is the case of plane Poiseuille flow (for $R > R_c$), the energy is focused at a point at the centre of the wave crest moving with the group velocity in a manner similar to, but different from, that proposed by Hocking *et al.* (1972) and Hocking & Stewartson (1972). Of particular interest is the case in which the nonlinearity accelerates the instability. In that case, after the amplitude has grown sufficiently, the effects of nonlinearity dominate both the dispersive and diffusive effects. The energy first focuses at one point in space. However, once the wave envelope becomes sharply peaked, the effects of dispersion and diffusion must once more become significant. It is shown in the appendix that these effects now modify the burst only in a small neighbourhood of it. Thus the energy eventually focuses, but only after a prescribed sequence of balances between nonlinearity, dispersion and diffusion.

In § 4, the nonlinear wave envelope equation is solved exactly in the case in which there is no diffusion. The results obtained are shown to correspond to asymptotic results developed in § 3.

The nonlinear evolution of aperiodic unstable waves is studied in order to understand further the instability of certain laminar flows. However, there are limits to the approach taken here. In particular, this analysis assumes that the spatial structure of the solution is approximately obtained from the linear problem, which does not appear to be the case experimentally for plane Poiseuille flow. Nonetheless a mathematical treatment of this more tractable problem is in order and can be considered as a step towards the understanding of the more difficult problem.

2. Formulation

In this section the type of amplitude equation appropriate for plane Poiseuille flow, equation (2.9), is shown to apply for a large class of nonlinear waves as was shown by Newell & Whitehead (1971). As one example, consider the following model initial-value problem:

$$u_t + L_x(u) = \epsilon k u |u|^2, \tag{2.1}$$

where L_x is a linear operator such that the linearization of (2.1) permits slightly amplified wavelike solutions with the dispersion relation $\omega(\alpha) = \omega_r(\alpha) + i\omega_i(\alpha)$.

The nonlinearity is weak, $0 < |\epsilon k| \ll 1$, and is in a form such as to allow direct modal transfer. A wavenumber α_m of maximum growth is assumed to exist, implying that $\omega_i'(\alpha_m) = 0$ and, in the simplest case, $\omega_i''(\alpha_m) < 0$. For the growth rate to be small, it is necessary that $0 < \omega_i(\alpha_m) \equiv \epsilon \ll 1$.

A solution to this weakly nonlinear initial-value problem is obtained by the techniques employed by Stewartson & Stuart (1971):

$$u = u_0 + \epsilon k u_1 + \dots, \tag{2.2}$$

where u_0 satisfies the linearized initial-value problem. For large x and t, the asymptotic behaviour of this solution can be shown by the usual method of steepest descents to be a slowly varying wave train:

$$u_0 \sim (Q(\bar{\alpha})/t^{\frac{1}{2}}) \exp[i\bar{\alpha}x - i\omega(\bar{\alpha})t], \tag{2.3}$$

where $\bar{\alpha}$ is the stationary point (assumed unique) of the phase

$$x/t = \omega'(\bar{\alpha}) \tag{2.4}$$

and where $$Q(\bar{\alpha}) = [i/2\omega''(\bar{\alpha})]^{\frac{1}{2}} \bar{F}(\bar{\alpha}),$$

where $\bar{F}(\alpha)$ is the Fourier transform of the initial condition. The growth rate is largest where $\bar{\alpha}$ is near α_m. If a non-dimensional quantity proportional to $\bar{\alpha} - \alpha_m$ is small, then it can be shown that

$$\left|\frac{\omega_i''(\alpha_m)}{\omega_i(\alpha_m)}\right|^{\frac{1}{2}} \left|\frac{x - \omega_r'(\alpha_m)t}{\omega''(\alpha_m)t}\right| \ll 1, \tag{2.5}$$

in which case

$$u_0 \sim \frac{Q(\alpha_m)}{t^{\frac{1}{2}}} \exp[i\alpha_m x - i\omega_r(\alpha_m)t] \exp[\omega_i(\alpha_m)t] \exp\left[\frac{(x - \omega_r'(\alpha_m)t)^2}{-2i\omega''(\alpha_m)t}\right]. \tag{2.6}$$

The algebraic decay associated with the dispersion of the initial energy retards the exponential growth due to the instability mechanism. The effect of diffusion is represented by the last exponential term in (2.6).

The analysis proceds by either evaluating u_1 asymptotically, in which case expansion (2.2) is shown, in §5, to become non-uniform when

$$\frac{|k|}{2t} \exp[2\omega_i(\alpha_m)t] \exp\left[\omega_i''(\alpha_m)\frac{(x - \omega_r'(\alpha_m)t)^2}{t|\omega''(\alpha_m)|^2}\right] = O(1),$$

or by allowing the amplitude to be a spatially and temporally slowly varying function. The latter procedure is now followed in order to compare these results with those obtained by Stewartson & Stuart (1971).

For large x and t, explicitly let the solution be the slowly varying wave train

$$u = A(X, \tau) \exp[i\alpha_m x - i\omega_r(\alpha_m)t], \tag{2.7}$$

where the slow variables are given by $X = \mu x$ and $\tau = \lambda t$, and where μ and λ will be determined. The equation for the amplitude A now follows from (2.1):

$$\lambda A_\tau + \mu\omega_r'(\alpha_m) A_X - \tfrac{1}{2}i\mu^2 \omega''(\alpha_m) A_{XX} - \omega_i(\alpha_m) A = \epsilon k A|A|^2, \tag{2.8}$$

where the neglected higher order dispersive and diffusive terms

$$\sum_{s=3}^{\infty} i \frac{(-i\mu)^s \omega^{(s)}(\alpha_m)}{s!} \frac{d^s A}{dX^s}$$

are of $O(\mu^3)$. In a co-ordinate system moving with the real group velocity, in which $\xi = X-(\mu/\lambda)\,\omega_r'(\alpha_m)\,\tau$, a non-trivial balance occurs if $\lambda = \mu^2 = \omega_i(\alpha_m)$. Consequently, the generalization of the amplitude equation (1.1) is derived:

$$A_\tau - a_2 A_{\xi\xi} = dA + kA|A|^2; \tag{2.9}$$

A must satisfy the initial condition (for $\tau \to 0$, $t \gg 1$)

$$A \to (\Delta/\tau^{\frac{1}{2}})\exp[-\xi^2/4a_2\tau], \tag{2.10}$$

where in (2.9) and (2.10)

$$\tau = \epsilon t, \quad \xi = \epsilon^{\frac{1}{2}}(x-a_{1r}t), \tag{2.11}$$

and where

$$\left.\begin{aligned}\epsilon &= \omega_i(\alpha_m) = \text{real growth rate of linearized theory,}\\ a_{1r} &= \omega_r'(\alpha_m) = \text{group velocity of linearized theory,}\\ a_2 &= a_{2r}+ia_{2i} = \tfrac{1}{2}i\omega''(\alpha_m),\\ \Delta &= \epsilon^{\frac{1}{4}}Q(\alpha_m),\\ d &= d_r + id_i.\end{aligned}\right\} \tag{2.12}$$

a_{2r} represents the diffusive effects of the spatial variation of the wave train, while a_{2i} represents the dispersive effects. Furthermore, the imaginary part of k corresponds to the familiar frequency modification effect common to nonlinear oscillators. d is the normalized amplification factor. In the model problem of this section $d = 1$, while for plane Poiseuille flow as derived by Stewartson & Stuart (1971) and Hocking *et al.* (1972) $d_r = 1$. This insignificant difference results from a different choice of the phase of the fundamental harmonic (compare equation (2.7) here with equation (3.3) in Stewartson & Stuart (1971)). In addition, (2.5) implies that

$$\frac{a_{2r}^{\frac{1}{2}}}{|a_2|}\frac{|\xi|}{\tau} \ll 1. \tag{2.13}$$

3. Asymptotic solution

Hocking *et al.* (1972) numerically solved (2.9) with initial condition (2.10) for the case in which the coefficients a_2, d, k and Δ are real. In particular, a_2 being real implies that the effect of the spatial modulations of the wave train is only to diffuse the amplitude. In this section, the problem is asymptotically solved for all ranges of the complex parameters a_2, d, k and Δ. Thus both the diffusive and dispersive effects of the spatial variations are allowed.

The amplitude is normalized according to

$$A = B\Delta \tag{3.1}$$

in order to introduce a small parameter ϵ_1, $0 < |\epsilon_1| \ll 1$. Then equation (2.9) governing the propagation of the wave envelope becomes

$$B_\tau - a_2 B_{\xi\xi} = dB + \epsilon_1 B|B|^2 \tag{3.2}$$

and must be solved with the matching condition

$$B \to \tau^{-\frac{1}{2}}\exp[-\xi^2/4a_2\tau] \quad \text{as} \quad \tau \to 0 \quad (t \gg 1), \tag{3.3}$$

where

$$\epsilon_1 = |\epsilon_1|e^{i\phi} = k|\Delta|^2 = \epsilon k|Q(\alpha_m)|^2. \tag{3.4}$$

d can be considered real since if it were not, then $\tilde{B} = Be^{-id_i\tau}$ would satisfy the same problem with $d = d_r$. Thus in what follows $d_i \equiv 0$.

Equation (3.2) is weakly nonlinear and hence the solution can be assumed in the form of a perturbation expansion

$$B = B_0 + \epsilon_1 B_1 + \dots. \tag{3.5}$$

B_0 satisfies the linearized problem, whose solution is known (Stuart 1971):

$$B_0 = (e^{d_r\tau}/\tau^{\frac{1}{2}}) \exp\left[-\xi^2/4a_2\tau\right]. \tag{3.6}$$

The equation for B_1 is

$$B_{1\tau} - a_2 B_{1\xi\xi} - d_r B_1 = \frac{e^{3d_r\tau}}{\tau^{\frac{3}{2}}} \exp\left[-\frac{\xi^2(2a_2^* + a_2)}{4|a_2|^2\tau}\right]. \tag{3.7}$$

Equation (3.7) implies that the inner expansion (3.5) is uniform when $\tau \leqslant O(1)$, as long as (2.13) is also satisfied. However, evaluating B_1 asymptotically for $\tau \gg 1$ using (3.7) gives

$$B_1 \sim \frac{e^{3d_r\tau}}{2d_r\tau^{\frac{3}{2}}} \exp\left[-\frac{\xi^2(2a_2^* + a_2)}{4|a_2|^2\tau}\right]. \tag{3.8}$$

Thus it follows that expansion (3.5) becomes disordered when $\tau \gg 1$ and

$$\frac{\epsilon_1 e^{2d_r\tau}}{2d_r\tau} \exp\left(-\frac{a_{2r}\xi^2}{2|a_2|^2\tau}\right) = O(1). \tag{3.9}$$

This is precisely when the exponential growth overcomes the algebraic decay of the steepest-descent solution. This suggests using the method of matched asymptotic expansions to obtain the solution when (3.9) is satisfied, as was done in a similar context by Newell, Lange & Aucoin (1970).

Let the inner variable be τ. The outer real variable T, implied by (3.9), is

$$T = \frac{|\epsilon_1|}{\tau} e^{2d_r\tau} \exp\left(-\frac{a_{2r}\xi^2}{2|a_2|^2\tau}\right). \tag{3.10}$$

Thus, for $T \ll 1$ but $\tau \gg 1$,

$$B \sim \frac{T^{\frac{1}{2}}}{|\epsilon|^{\frac{1}{2}}} \exp\left(i\frac{a_{2i}\xi^2}{4|a_2|^2\tau}\right)\left[1 + \frac{Te^{i\phi}}{2d_r} + \dots\right]. \tag{3.11}$$

This implies that the outer expansion should be of the form

$$B = \frac{1}{|\epsilon_1|^{\frac{1}{2}}} \exp\left(i\frac{a_{2i}\xi^2}{4|a_2|^2\tau}\right) f(T) + \dots. \tag{3.12}$$

$f(T)$ will give the *leading-order* behaviour of the solution for $T = O(1)$. The equation for $f(T)$ is to be determined, solved and its solution matched to equation (3.11). The diffusive and dispersive term $B_{\xi\xi}$ can be shown to have no effect on the equation for $f(T)$ as long as (2.13) is satisfied. The equation for $f(T)$

$$2d_r T\, df/dT = d_r f + e^{i\phi} f^2 f^* \tag{3.13}$$

has the general solution

$$f = \left(\frac{cTd_r/\cos\phi}{1 - cT}\right)^{\frac{1}{2}} e^{i\theta(T)}, \tag{3.14}$$

where
$$\theta(T) = -\tfrac{1}{2}\tan\phi \ln(1-cT)+\theta_0, \tag{3.15}$$
and where c and θ_0 are arbitrary real constants. By matching this solution to (3.11), it is seen that
$$\theta_0 = 0, \quad c = \cos\phi/d_r. \tag{3.16}$$
(These results must be modified if $\phi = \frac{1}{2}\pi$, in which case
$$f = b^{\frac{1}{2}}T^{\frac{1}{2}}\exp[i(bT/2d_r+\theta_0^*)],$$
where, by matching, $b = 1$ and $\theta_0^* = 0$. This corresponds to the special case in which the only nonlinear effect is the well-understood frequency modification. However, in the problems of interest $\phi \neq \frac{1}{2}\pi$.) Consequently the leading-order behaviour of B, when $T \geqslant O(1)$ (i.e. $\tau \gg 1$), is
$$B = \frac{1}{|\epsilon_1|^{\frac{1}{2}}}\exp\left(i\frac{a_{2i}\xi^2}{4|a_2|^2\tau}\right)\left(\frac{T}{1-\dfrac{\cos\phi}{d_r}T}\right)^{\frac{1}{2}}\exp\left[-\frac{i}{2}\tan\phi\ln\left(1-\frac{\cos\phi}{d_r}T\right)\right]. \tag{3.17}$$

Equation (3.17) indicates that the amplitude reaches an equilibrium value as $T \to \infty$ if $\cos\phi/d_r < 0$. In this case as $T \to \infty$
$$B \to \frac{\exp(ia_{2i}\xi^2/4|a_2|^2\tau)}{|\epsilon_1|^{\frac{1}{2}}}\left(\frac{-d_r}{\cos\phi}\right)^{\frac{1}{2}}\exp\left[-i\frac{\tan\phi}{2}\ln\left(1-\frac{\cos\phi}{d_r}T\right)\right], \tag{3.18a}$$
or, as is more meaningful,
$$BB^* \to -d_r/|\epsilon_1|\cos\phi. \tag{3.18b}$$
In order for $T \to \infty$, only the case $d_r > 0$ is relevant and hence $\cos\phi < 0$. Thus $k_r < 0$. The nonlinear effects are stabilizing as in the corresponding problem without spatial modulations.

In the case in which the cubic nonlinearity is destabilizing (for example, plane Poiseuille flow for $R > R_c$), $\cos\phi/d_r > 0$ (i.e. $d_r > 0$, $k_r > 0$). Thus the solution (3.17) becomes infinite when
$$T = d_r/\cos\phi,$$
i.e.
$$\frac{|\epsilon_1|}{\tau}e^{2d_r\tau}\exp\left(-\frac{a_{2r}\xi^2}{2|a_2|^2\tau}\right) = \frac{d_r}{\cos\phi}. \tag{3.19}$$

Equation (3.19) determines the time and position of the first singularity. The energy is focused at $\xi = 0$, the centre of the wave of the largest linear growth rate moving with its group velocity. The amplitude goes to infinity first when $\tau = \tau_0 \gg 1$ such that
$$\frac{|\epsilon_1|}{\tau_0}e^{2d_r\tau_0} = \frac{d_r}{\cos\phi}. \tag{3.20}$$

Note that this critical time is independent of a_2, the parameter representing the diffusive and dispersive effects. As is also clear, the smaller the initial nonlinearity $|\epsilon_1| = |k||\Delta|^2$, the longer it takes for the amplitude to become singular. Furthermore, this analysis obtains analytic (but asymptotic) results for the cases numerically studied by Hocking *et al.* (1972). There, because of the computational nature of the problem, real coefficients were chosen: $d_r = k = a_2 = 1$. Thus,

Initial amplitude	Time obtained from asymptotic formula	Time obtained from numerical results
Δ	τ_0 ($\tau_0 \gg 1$)	τ_0
0·01	5·45	4·23
0·095	2·88	2·95

TABLE 1. Time of amplitude singularity: comparison between asymptotic formula and numerical results of Hocking *et al.* (1972)

from (3.4) and (3.20), it follows that $|\epsilon_1| = |\Delta|^2$, $\cos\phi = 1$ and τ_0 is determined from

$$(|\Delta|^2/\tau_0)\, e^{2\tau_0} = 1.$$

The values of τ_0 corresponding to $\Delta = 0{\cdot}01$ and $\Delta = 0{\cdot}095$ agree reasonably well with those Hocking *et al.* (1972) obtained by numerically integrating the initial-value problem. The comparison is shown in table 1.

Since the work in this section only analyses the leading-order term in the far field, the diffusive and dispersive effects have been neglected. The example in §4 suggests that the dispersion in the far field by itself will not significantly alter the behaviour of the solution near the singularity. In the appendix, a systematic perturbation expansion is introduced which considers both dispersion and diffusion. It is shown that, although the resulting expansion is not uniformly valid, the leading-order term correctly gives the structure, time and position of the singularity. Consequently, for example, the structure of the solution near the singularity is (for details see the appendix)

$$B = \left(\frac{d_r}{|\epsilon_1|\cos\phi}\right)^{\frac{1}{2}} \left[2d_r(\tau_0-\tau)+\frac{a_{2r}\xi^2}{2|a_2|^2\tau_0}\right]^{-\frac{1}{2}} \times \exp\left[-\frac{i}{2}\tan\phi\ln\left(2d_r(\tau_0-\tau)+\frac{a_{2r}\xi^2}{2|a_2|^2\tau_0}\right)\right]. \quad (3.21)$$

The effect of the spatial modulation of the initial wave envelope is to allow the diffusion and dispersion to retard the growth due to the nonlinear amplification. This only moderately weakens the amplification and eventually the nonlinearity is dominant. Thus in this model a singularity is reached in a finite time, but this time is substantially longer than it is in the case without spatial modulations. The singularity can be prevented only by including higher order nonlinear effects neglected in the formulation of this problem (equation (2.1) or (2.9)). None the less, it should be expected that near this critical time all the neglected nonlinear terms will become equally important. Likewise all the higher harmonics should become significant and thus this critical time is perhaps an indication of the breakdown of the laminar flow.

4. Non-diffusive case

The case $a_{2r} = 0$ corresponds to an inflexion point in the growth rate; $\omega_i''(\alpha_m) = 0$. (For the growth rate to be a local maximum, $\omega_i'''(\alpha_m) = 0$.) The problem is non-diffusive; the effect of the spatial modulations is only dispersive in nature. In this case, the results of the previous sections can be applied in a straightforward manner. However, some insight is obtained by considering this case, since then the following exact solution to (3.2) exists:

$$B(\xi,\tau) = \exp\left(-\frac{\xi^2}{4a_2\tau}\right)\frac{e^{d_r\tau}}{\tau^{\frac{1}{2}}}\frac{e^{i\theta}}{\left(c-2|\epsilon_1|\cos\phi\int_1^\tau \frac{e^{2d_r w}}{w}\,dw\right)^{\frac{1}{2}}}, \tag{4.1}$$

with
$$\theta = |\epsilon_1|\sin\phi\int_1^\tau |B|^2\,dt+\theta_0, \tag{4.2}$$

where c and θ_0 are arbitrary constants. This solution is similar to one obtained by Benney & Newell (1967) for the case of a neutrally stable non-diffusive system $d_r = \cos\phi = 0$ (i.e. $d_r = k_r = 0$). The integral in (4.2) can be evaluated, yielding

$$\theta = -\frac{\tan\phi}{2}\ln\left[\frac{1}{c}\left(c-2|\epsilon_1|\cos\phi\int_1^\tau \frac{e^{2d_r w}}{w}\,dw\right)\right]+\theta_0. \tag{4.3}$$

If
$$\left|2\epsilon_1\cos\phi\int_1^\tau \frac{e^{2d_r w}}{w}\,dw\right| \ll c \tag{4.4}$$

as $\tau\to 0$ ($t \gg 1$), then fortuitously $B(\xi,\tau)$ given by (4.1) also satisfies the initial conditions (3.3) if

$$e^{i\theta_0}/c^{\frac{1}{2}} = 1 \quad (\text{i.e.}\quad \theta_0 = 0,\quad c = 1). \tag{4.5}$$

Therefore (4.1) is the desired solution in the case $a_{2r} = 0$ if the inequality (4.4) is satisfied.

To investigate the inequality (4.4), it is convenient to rewrite the integral as

$$\int_1^\tau \frac{e^{2d_r w}}{w}\,dw = \int_1^\tau \frac{e^{2d_r w}-1}{w}\,dw+\ln\tau. \tag{4.6}$$

The criterion for the initial conditions to be satisfied is now seen to be

$$|2\epsilon_1\cos\phi\ln\tau| \ll 1 \tag{4.7}$$

in the limit $\tau\to 0$ ($t\gg 1$). Since $\tau = \epsilon t$, the inequality (4.7) holds as long as ϵ is not transcendentally smaller than $|\epsilon_1|$. This logarithmic singularity as $\tau\to 0$ emphasizes the point that the initial conditions (3.3) must be treated in the context of the method of matched asymptotic expansions, that is, for $\tau\to 0$ ($t\gg 1$).

In particular, the asymptotic expansion of $B(\xi,\tau)$ for large τ is obtained from (4.1) and (4.3) by noting that for large τ

$$\int_1^\tau \frac{e^{2d_r w}}{w}\,dw \sim \frac{e^{2d_r\tau}}{2d_r\tau}. \tag{4.8}$$

Therefore for large τ

$$B(\xi,\tau) \sim \exp\left(-\frac{\xi^2}{4a_2\tau}\right)\frac{e^{d_r\tau}}{\tau^{\frac{1}{2}}}\frac{e^{i\theta}}{\left(1-\dfrac{|\epsilon_1|\cos\phi\, e^{2d_r\tau}}{d_r\tau}\right)^{\frac{1}{2}}} \tag{4.9}$$

and
$$e^{i\theta} \sim \exp\left[id_i\tau-i\frac{\tan\phi}{2}\ln\left(1-\frac{|\epsilon_1|\cos\phi\, e^{2d_r\tau}}{d_r\tau}\right)\right], \tag{4.10}$$

which agree with the previously derived asymptotic result, equation (3.18) in the case $a_{2r} = 0$. Furthermore, if $d_r > 0$ and $\cos\phi < 0$, then the asymptotic expansion (4.9) can be simplified when

$$-\frac{|\epsilon_1|\cos\phi}{d_r}\frac{e^{2d_r\tau}}{\tau} \gg 1, \tag{4.11}$$

in which case
$$B(\xi,\tau) \sim \exp\left(-\frac{\xi^2}{4a_2\tau}\right)\frac{e^{i\theta}}{(-|\epsilon_1|\cos\phi/d_r)^{\frac{1}{2}}} \tag{4.12}$$

and $e^{i\theta}$ is still given asymptotically by (4.10). Equations (4.10)–(4.12) also correspond directly to equations (3.9) and (3.18) in the asymptotic analysis when $a_{2r} = 0$. Alternatively, if $\cos\phi/d_r > 0$, equation (4.1) indicates that $B(\xi,\tau)$ becomes infinite when

$$1 = 2|\epsilon_1|\cos\phi\int_1^\tau \frac{e^{2d_r w}}{w}\,dw, \tag{4.13}$$

which, since $0 < |\epsilon_1| \ll 1$, occurs at some large value of τ. This large value of τ, obtained asymptotically using (4.8), agrees with the value obtained by the asymptotic analysis of the previous sections, equation (3.19) in the limit $a_{2r} = 0$.

Here all the results of the asymptotic analysis in § 3 have been verified from the exact solution for the case in which the problem is non-diffusive.

5. Large-time behaviour without slow variations

In §§ 2–4 the asymptotic solution for large time was obtained by combining the method of slow variations with the method of matched asymptotic expansions. In this section, it is suggested that a matching procedure by itself is simpler.

It is recalled that u_0 can be calculated and evaluated asymptotically for large t (from equation (2.3) and under further approximations equation (2.6)). Continuing with the perturbation expansion (2.2),

$$u_{1t} + L_x(u_1) = u_0|u_0|^2. \tag{5.1}$$

u_1 itself is not needed, only its asymptotic behaviour for large x and t, such that (2.5) is still satisfied. This is found to be

$$u_1 \sim |Q|^2 Q\frac{\exp\{3\omega_i(\alpha_m)t\}}{2\omega_i(\alpha_m)t^{\frac{3}{2}}}\exp\{i[\alpha_m x - \omega_r(\alpha_m)t]\} \times\exp\left\{i\frac{(x-\omega_r'(\alpha_m)t)^2}{2t|\omega''(\alpha_m)|^2}(2\omega''^*(\alpha_m)-\omega''(\alpha_m))\right\}. \tag{5.2}$$

Thus the expansion (2.2) is non-uniform when

$$\frac{\epsilon k|Q|^2\exp\{2\omega_i(\alpha_m)t\}}{2\omega_i(\alpha_m)t}\exp\left[\frac{(x-\omega_r'(\alpha_m)t)^2}{t|\omega''(\alpha_m)|^2}\omega_i''(\alpha_m)\right] = O(1). \tag{5.3}$$

Equation (5.3) implies that the outer variable T is again given by (3.10) (here $d_r = 1$). Proceeding directly to matching, the inner solution to two terms (u_0 and u_1) in terms of the outer variable to one term is, from (2.5), (3.4), (5.2) and (5.3),

$$u \sim \frac{Q e^{\frac{1}{2}i\phi}}{|Q|k^{\frac{1}{2}}}\exp[i\alpha_m x - i\omega_r(\alpha_m)t]\exp\left[\frac{i\omega_r''(\alpha_m)}{2t|\omega''(\alpha_m)|^2}(x-\omega_r'(\alpha_m)t)^2\right]T^{\frac{1}{2}}\left(1+\frac{e^{i\phi}}{2}T\right). \tag{5.4}$$

Thus the leading-order behaviour of the outer solution is given by

$$u = \frac{Q\, e^{\frac{1}{2}i\phi}}{|Q|\, k^{\frac{1}{2}}} \exp\left[i\alpha_m x - i\omega_r(\alpha_m)\, t\right] \exp\left[\frac{i\omega_r''(\alpha_m)}{2t|\omega''(\alpha_m)|^2}(x-\omega_r'(\alpha_m)\, t)^2\right] f(T). \quad (5.5)$$

Using (3.1) and (3.4), it is seen that (5.5) corresponds exactly to (3.12). Thus (3.13) will again result, yielding again the solution given by (3.14)–(3.16) since the matching condition implied by (5.4) is equivalent to that implied previously by (3.11).

Hence the results of §3 hold without the necessity of the introduction of the concepts of slow variations. Spatial and temporal modulations of a uniform wave train are obtained using only the method of matched asymptotic expansions. This is not to say that (3.2) is incorrect, only that it is unnecessary in order to determine the long-time asymptotic behaviour in these types of problems.

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Appendix. Higher order terms

For slightly unstable nonlinear wave systems when $d_r > 0$ and $\cos\phi > 0$, the analysis in §3 showed that, for the leading-order term in the far field, a singularity is reached at a finite value of T. This corresponds to a singularity first occurring at $\tau = \tau_0$ for one value of $\xi = \xi_0 = 0$. For that leading-order term, the dispersive and diffusive effects were neglected near the singularity. Here, higher order terms which include these effects are calculated. It is shown that the resulting expansion is not uniformly valid. The diffusive and dispersive terms become important near the singularity. However, the structure, time and position of the singularity are given uniformly by the leading-order term.

The structure of this singularity is not consistent with that obtained by Hocking *et al.* (1972) in the case when the quantities are real or by Hocking & Stewartson (1972) in the general case. The present author believes that the expansion near the singularity they developed is in error, as it is not uniformly valid in a finite neighbourhood of $\xi = 0$ as $\tau \to \tau_0$. In particular, as $\tau \to \tau_0$ such that $\zeta \to \infty$, equation (2.9) of Hocking & Stewartson (1972) is disordered owing to the prevalence of the logarithmic terms. Hocking & Stewartson (1972) proposed another possible structure of the singularity. However, it does not apply in most cases. It is similar to the type of singularity derived here, although different in certain important ways. Specifically, the structure of the singularity obtained in this appendix is valid in all cases.

The leading-order term, equation (3.17), is singular when $T = d_r/\cos\phi$. Expanding in a Taylor series around the first item $\tau = \tau_0$ and corresponding position $\xi = \xi_0 = 0$, it is found that to leading order

$$1 - \frac{\cos\phi}{d_r} T \approx 2d_r(\tau_0 - \tau) + \frac{a_{2r}}{|a_2|^2}\frac{\xi^2}{2\tau_0},$$

where τ_0 is large and is given by (3.20). Note that, if $a_{2r} = 0$, the analysis of § 4 would apply. Thus these higher order effects are only diffusive in nature.

In order to obtain the higher order terms near the singularity in a systematic fashion, (3.2) should be rescaled based on the leading-order term

$$B = \left(\frac{d_r}{\cos\phi}\right)^{\frac{1}{2}} \frac{\bar{B}}{|\epsilon_1|^{\frac{1}{2}}}, \quad \xi = \frac{|a_2|\,(2\tau_0)^{\frac{1}{2}}}{(a_{2r})^{\frac{1}{2}}}\,\bar{\xi}, \quad \tau_0 - \tau = -\bar{\tau}/d_r. \tag{A 1}$$

The resulting equation for $\bar{B}$ is

$$\bar{B}_{\bar{\tau}} = \bar{B} + (1+i\tan\phi)\,\bar{B}|\bar{B}|^2 + \lambda B_{\bar{\xi}\bar{\xi}},$$

where $$\lambda = a_2 a_{2r}/2|a_2|^2 d_r \tau_0 \quad \text{and} \quad |\lambda| \ll 1$$

(note that the real part of λ is greater than 0). Since near the singularity the cubic term will dominate the linear term, it is convenient instead to consider

$$\bar{B}_{\bar{\tau}} = (1+i\tan\phi)\,\bar{B}|\bar{B}|^2 + \lambda\bar{B}_{\bar{\xi}\bar{\xi}}. \tag{A 2}$$

In this form the leading-order equation for $|\lambda| \ll 1$ has the solution

$$\bar{B}_0 = (-2\bar{\tau}+\bar{\xi}^2)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}i\tan\phi \ln(-2\bar{\tau}+\bar{\xi}^2)\right], \tag{A 3}$$

corresponding to (3.17). A similarity solution of (A 2) exists in the form

$$\bar{B} = (-2\bar{\tau})^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}i\tan\phi \ln(-2\bar{\tau}+\bar{\xi}^2)\right] F(\zeta), \tag{A 4}$$

where $$\zeta = \bar{\xi}/(-2\bar{\tau})^{\frac{1}{2}}. \tag{A 5}$$

Thus, after some algebraic manipulations,

$$\zeta F_\zeta + F + \frac{i\tan\phi}{1+\zeta^2}F - (1+i\tan\phi)\,F|F|^2 = \lambda\left[F_{\zeta\zeta} - \frac{2i\tan\phi\,\zeta F_\zeta}{1+\zeta^2} + i\tan\phi\,\frac{\zeta^2(1+i\tan\phi)-1}{(1+\zeta^2)^2}F\right]. \tag{A 6}$$

The outer expansion is a power series in $|\lambda|$:

$$F = \sum_{n=0}^{\infty} |\lambda|^n f_n(\zeta). \tag{A 7}$$

The zeroth-order equation

$$\zeta f_{0\zeta} + f_0 + \frac{i\tan\phi}{1+\zeta^2} f_0 = (1+i\tan\phi) f_0|f_0|^2 \tag{A 8}$$

is satisfied by $$f_0 = 1/(1+\zeta^2)^{\frac{1}{2}}, \tag{A 9}$$

which corresponds to the previous leading-order term, equation (A 3). The higher order terms are determined from equations of the form

$$\zeta f_n + f_n + \frac{i\tan\phi}{1+\zeta^2} f_n - \frac{(1+i\tan\phi)}{1+\zeta^2}(2f_n + f_n^*) = g_n,$$

where, for each n, g_n is a functional of the previously determined functions $f_0, \ldots, f_{n-1}$. The solution to this linear problem can be obtained in quadrature:

$$f_n = \tfrac{1}{2}(1+\zeta^2)^{-\frac{3}{2}}\left[I_{2n}(1+2\zeta^2 - i\tan\phi) - I_{2n}^*(1+i\tan\phi) + \zeta^2(I_{1n} + I_{1n}^*)(1+i\tan\phi)\right], \tag{A 10}$$

where $$I_{1n} = \int^{\zeta} \frac{(1+x^2)^{\frac{1}{2}} g_n(x)}{x^3}\,dx, \quad I_{2n} = \int^{\zeta} \frac{(1+x^2)^{\frac{1}{2}} g_n(x)}{x}\,dx$$

(the arbitrary constants of integration must be real). In the non-dispersive case discussed by Hocking *et al.* (1972) (all quantities are real and thus $\phi = 0$), this reduces to the result they obtained in a slightly different context:

$$f_n = \frac{\zeta^2}{(1+\zeta^2)^{\frac{3}{2}}} \int^{\zeta} \frac{(1+x^2)^{\frac{3}{2}} g_n(x)}{x^3}\,dx.$$

It should be noted that in the approach taken here the functional form of g_n is different from theirs.

The higher order terms must be computed asymptotically in order to show that the outer expansion (A 7) becomes disordered. Then the method of matched asymptotic expansions is used to obtain the inner expansion. For the zeroth-order term,

$$f_0 = \begin{cases} 1-\frac{1}{2}\zeta^2+\frac{3}{8}\zeta^4+O(\zeta^6) & \text{as} \quad \zeta \to 0, \\ 1/\zeta + O(1/\zeta^3) & \text{as} \quad \zeta \to \infty. \end{cases} \tag{A 11}$$

It can be shown that the higher order terms are well ordered as $\zeta \to \infty$. However, as $\zeta \to 0$, the outer expansion is disordered as is implied from the following results for $\zeta \to 0$, in which λ complex ($\lambda \neq \lambda^*$) yields the dispersive and diffusive case and λ real ($\lambda = \lambda^*$) the non-dispersive case:

$$f_1 = \begin{cases} [(\lambda^*-\lambda)/2|\lambda|]\,(1+\tan^2\phi)\ln\zeta + O(1) & \text{for} \quad \lambda \text{ complex}, \\ \frac{1}{2}(1+i\tan\phi) + (1+i\tan\phi)\,(3-\tan^2\phi)\,\zeta^2\ln\zeta & \\ \qquad + \frac{1}{4}\zeta^2(-3-i\tan\phi+2i\tan^3\phi)+\ldots & \text{for} \quad \lambda \text{ real}, \end{cases}$$

$$f_2 = \begin{cases} [(\lambda^*-\lambda)/|\lambda|]\,O(1/\zeta^2) & \text{for} \quad \lambda \text{ complex}, \\ \ln\zeta[(1+i\tan\phi)\,(\tan^2\phi-3)+i\tan\phi(1+\tan^2\phi)] + O(1) & \text{for} \quad \lambda \text{ real}, \end{cases}$$

$$f_{n+2} = \begin{cases} [(\lambda^*-\lambda)/|\lambda|]\,O(1/\zeta^{2(n+1)}) & \text{for} \quad \lambda \text{ complex}, \\ \dfrac{(-1)^{n+1}}{\zeta^{2n}}\dfrac{(2n-1)!}{2^n}\left[\dfrac{1}{(n+1)!}(1+i\tan\phi)\,(\tan^2\phi-3) + \dfrac{2i\tan\phi(1+\tan^2\phi)}{n!}\right] & \text{for} \quad \lambda \text{ real}, \end{cases} \quad n \geqslant 1. \tag{A 12}$$

In the case in which λ is real, the logarithmic singularity in the term f_1 vanishes if $\tan^2\phi = 3$. The higher order terms are still disordered. Hocking & Stewartson (1972) showed that the higher order terms (in both the case in which λ is real and when λ is complex) are not necessarily disordered. This was done by considering a similarity solution of (A 2) which is slightly more general than (A 4). In order to make their solution regular at the centre of the 'burst' ($\zeta \to 0$), they insisted that the expansion be well-ordered there. However, in that manner they obtained a solution only if a certain relationship existed between $\tan\phi$ and λ, which, for example, when λ is real is equivalent to $\tan^2\phi = 3$ derived above to leading order in powers of $|\lambda|$. In the present work, a similar limitation in parameter space is not necessary; solutions are obtained for all values of $\tan\phi$ and all complex values of λ

as long as $|\lambda| \ll 1$. Here it is not required that the expansion be well-ordered. Instead, the singularity as $\zeta \to 0$, characteristic of the disordering of an outer expansion, will be analysed by a boundary layer in the ζ variable.

The form of the breakdown of the outer solution depends on whether λ is real or not. In either case it is seen that the inner variable is

$$s = \zeta/|\lambda|^{\frac{1}{2}} = \bar{\xi}/[-2|\lambda|\bar{\tau}]^{\frac{1}{2}}. \tag{A 13}$$

The inner solution is given by (A 4), where the inner equation for $F(s)$ is

$$sF_s + F + \frac{i\tan\phi}{1+|\lambda|s^2}F = (1+i\tan\phi)F|F|^2 + \frac{\lambda}{|\lambda|}F_{ss} + \lambda i \tan\phi\left[\frac{-2sF_s}{1+|\lambda|s^2} + \frac{|\lambda|s^2(1+i\tan\phi)-1}{(1+|\lambda|s^2)^2}F\right]. \tag{A 14}$$

In the case discussed by Hocking *et al.* (1972), where λ is real and positive, $\phi = 0$ and F is real, this reduces to

$$sF_s + F = F^3 + F_{ss},$$

which is an equation they obtained in a slightly different context. In their appendix, S. N. Brown gave a non-existence theorem for this equation with the boundary conditions $F'(0) = 0$ and $F(\infty) = 0$. Here $F'(0) = 0$, but $F(s)$ for large s must match to the outer solution for small ζ (this will certainly imply $F(\infty) \neq 0$).

The asymptotic expansion of the outer solution in terms of the inner variable for λ complex ($\lambda \neq \lambda^*$) is

$$F = 1 + \tfrac{1}{4}(\lambda^* - \lambda)\ln|\lambda|\,(1+\tan^2\phi) + |\lambda|\left[-\tfrac{1}{2}s^2 + \bar{O}(1) + \frac{\lambda^*-\lambda}{2|\lambda|}(1+\tan^2\phi)\ln s + \bar{O}\left(\frac{1}{s^2}\right) + \bar{O}\left(\frac{1}{s^4}\right) + \dots\right], \tag{A 15a}$$

while for λ real ($\lambda = \lambda^*$)

$$F = 1 + \tfrac{1}{2}\lambda(1 + i\tan\phi - s^2) + \tfrac{1}{2}\lambda^2\ln\lambda\,[(1+i\tan\phi)(\tan^2\phi - 3)(1-s^2) + i\tan\phi(1+\tan^2\phi)] + \lambda^2\left[\tfrac{3}{8}s^4 - (1+i\tan\phi)(\tan^2\phi-3)s^2\ln s + \tfrac{1}{4}s^2(-3 - i\tan\phi + 2i\tan^3\phi) + \bar{O}(1) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\zeta^{2n}}\frac{(2n-1)!}{2^n} \times\left(\frac{(1+i\tan\phi)(\tan^2\phi-3)}{(n+1)!} + \frac{2i\tan\phi(1+\tan^2\phi)}{n!}\right)\right] + O(\lambda^3\ln\lambda). \tag{A 15b}$$

(In these expressions the symbol $\bar{O}(s^n) = c_n s^n$ for some constant c_n independent of s which has not been determined, but can be.) The inner solution is thus of the form

$$F(s) = 1 + \tfrac{1}{4}(\lambda^* - \lambda)\ln|\lambda|(1+\tan^2\phi) + |\lambda|F^{(2)}(s) + O(|\lambda|^2\ln|\lambda|) \tag{A 16a}$$

for λ complex ($\lambda \neq \lambda^*$) and

$$F(s) = 1 + \tfrac{1}{2}\lambda(-s^2 + 1 + i\tan\phi) + \tfrac{1}{2}\lambda^2\ln\lambda[(1+i\tan\phi)(\tan^2\phi-3)(1-s^2) + 2i\tan\phi(1+\tan^2\phi)] + \lambda^2F^{(4)}(s) + O(\lambda^3\ln\lambda) \tag{A 16b}$$

for λ real ($\lambda = \lambda^*$).

By substituting these expressions into (A 14), the equations for the unknown functions result. For the case $\lambda \neq \lambda^*$,

$$-s\frac{dF^{(2)}}{ds}+(1+i\tan\phi)(F^{(2)}+F^{(2)*})+\frac{\lambda}{|\lambda|}\frac{d^2F^{(2)}}{ds^2}=i\tan\phi\left(-s^2+\frac{\lambda}{|\lambda|}\right), \quad \text{(A 17}a)$$

whose solution, though presumably quite complicated, certainly exhibits no singularities as $s \to 0$ (and will be able to match as $s \to \infty$). On the other hand if λ is real ($\lambda = \lambda^*$), then the problem of interest is simpler:

$$s\frac{dF^{(4)}}{ds}-(1+i\tan\phi)(F^{(4)}+F^{(4)*})-\frac{d^2F^{(4)}}{ds^2}$$
$$= s^4(\tfrac{3}{4}-\tfrac{3}{4}i\tan\phi)+s^2(-\tfrac{3}{2}+4i\tan\phi-\tan^2\phi)+\tfrac{1}{4}(1+i\tan\phi)(3+\tan^2\phi). \quad \text{(A 17}b)$$

The general solution of (A 17b) (which is not exponentially large in order to match to the outer solution) is

$$\begin{aligned} F^{(4)} = {}& \tfrac{3}{8}s^4+(1+i\tan\phi)(-3+\tan^2\phi) \\ & \times\left\{\tfrac{13}{8}+(1-s^2)\int_0^s e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}y^2}\,dy\,dx+s\,e^{\frac{1}{2}s^2}\int_s^\infty e^{-\frac{1}{2}x^2}\,dx\right\} \\ & +i\tan\phi(1+\tan^2\phi)\left[\int_0^s e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}y^2}\,dy\,dx+\frac{s^2}{2}\right] \\ & -\tfrac{1}{4}\tan^2\phi(1+i\tan\phi)+c_1(1+i\tan\phi)(s^2-1)+ic_2, \end{aligned} \quad \text{(A18)}$$

where c_1 and c_2 are arbitrary real constants, chosen so that the solution can be matched. Thus, explicitly, $F^{(4)}$ is not singular as $s \to 0$.

Hence, even though the outer expansion is not uniformly valid, the first term of that expansion is uniformly valid. Thus, no matter how $\bar{\xi} \to 0$ and $\bar{\tau} \to 0$,

$$\bar{B} \sim (-2\bar{\tau}+\bar{\xi}^2)^{-\frac{1}{2}}\exp\left[-\tfrac{1}{2}i\tan\phi\ln(-2\bar{\tau}+\bar{\xi}^2)\right]. \quad \text{(A 19)}$$

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